Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(00)00017-4

J. Appl. Maths Mechs, Vol. 63, No. 6, pp. 989–992, 1999

0021-8928/99/$—see front matter

THE DUAL NATURE OF THE TRANSVERSE VIBRATIONS OF AN ELASTIC ROD†

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(*Received* 12 *May* 1999)

The transverse vibrations of an elastic rod, to one of which displacements are applied while the other end is free, are investigated. It is assumed that the propagation velocity of the perturbations in the rod is finite. The unperturbed part performed rotational motion around the centre line. The angle of rotation is expressed by the angle of curvature of the centre line of the perturbed part of the rod. Two types of elastic vibrations are obtained: (1) the rod vibrates elastically due to displacements applied at the end, and (2) when performing rotational motion elastic vibrations and additional forces occur in the rod due to elasticity [1].

Consider the transverse vibrations of an elastic rod of length l, when a displacement $A\sin\omega t$ is applied to one end, while the other end is free. We will consider a solution of the form

$$w(t, y) = w_1(t, y) + A\sin\omega t$$

The function $w_1(t, y)$ must satisfy the following inhomogeneous equation with boundary and initial conditions

$$\frac{\partial^2 w_1}{\partial t^2} + c^2\frac{\partial^4 w_1}{\partial y^4} = A\omega^2\sin\omega t \quad \left(c^2 = \frac{EJ}{\rho F}\right) \tag{1}$$

$$w_1(t,0) = 0, \quad \frac{\partial w_1(t,0)}{\partial y} = 0$$

$$\frac{\partial^2 w_1(t,l)}{\partial y^2} = 0, \quad \frac{\partial^3 w_1(t,l)}{\partial y^3} = 0 \tag{2}$$

$$w_1(0,y) = 0, \quad \dot{w}_1(0,y) = -A\omega \tag{3}$$

Here c is a constant coefficient, E is the modulus of elasticity of the material, J is the moment of inertia, ρ is the density and F is the cross-section area of the rod.

The solution of Eq. (1) with the given right-hand side will be sought in the form of a sum

$$w_1(t,y) = \sum_{m=1}^{\infty}\theta_m(t)Z_m(y) \tag{4}$$

where $Z_m(y)$ is the mth natural mode of oscillation, $\theta_m(t)$ is the coefficient of dynamic increment in the mth natural mode of oscillation and $m = 1, 2, \ldots$.

The eigenfunctions $Z_m(y)$ are constructed taking into account their orthogonality with weight $n(y)$ in the section [0, 1]. After separating the variables, Eq. (1) reduces to a system of ordinary differential equations which contains the natural frequency of the mth mode of oscillation $k_m = c\lambda_m^2$. The general solutions of this system corresponding to boundary and initial conditions (2) and (3) are well known and have the form

$$Z_m(y) = A_m[U(\mu_m y/l) - \beta_m V(\mu_m y/l)]$$

$$\theta_m(t) = C_m\sin k_m t + \frac{1}{k_m}\int_0^t \Phi_m(\tau)\sin k_m(t-\tau)d\tau$$

$$C_m = -\frac{A\omega}{k_m\alpha_m^2}\int_0^l Z_m(y)dy, \quad \Phi_m(t) = -\omega k_m C_m\sin\omega t$$

†*Prikl. Mat. Mekh.* Vol. 63, No. 6, pp. 1055–1058, 1999.

$$\beta_m = \frac{\operatorname{sh}\mu_m - \sin\mu_m}{\operatorname{ch}\mu_m + \cos\mu_m}, \quad \mu_m = \lambda_m l$$

where μ_m are the roots of the equation ch μ cos μ + 1 = 0.

In the case considered the weight $n(y) = 1$, and $U(\cdot)$ and $V(\cdot)$ are Krylov functions (2).

The constants A_m are found from the conditions for the eigenfunctions to be orthogonal.

After reduction, we can represent the solution of this problem in the form

$$w_1(t,y) = \sum_{m=1}^{\infty} \frac{A\omega\gamma_m\beta_m\bar{Z}_m(y)}{\lambda_m(k_m^2 - \omega^2)}(\omega\sin\omega t - k_m\sin k_m t) \tag{5}$$

$$\gamma_m = \frac{4(\operatorname{ch}\lambda_m l + \cos\lambda_m l)^2}{l\operatorname{sh}^2\lambda_m l\sin^2\lambda_m l}, \quad \bar{Z}_m(y) = \frac{Z_m(y)}{A_m}$$

It can be seen that the vibrations consist of two parts: forced vibrations, proportional to sin ωt, and free vibrations proportional to sin $k_m t$.

When the frequency of the perturbing force is approximately equal to one of the natural vibration frequencies, a resonance occurs.

The linear integro-differential equation of motion and the equation of the elastic transverse vibrations of the rod in the case of high bending stiffness have the form [1]

$$\rho F\int_0^l y(y\ddot{\varphi} + w_2)dy = M(t) - \rho gF\int_0^l (y\cos\varphi - w_2\sin\varphi)dy \tag{6}$$

$$\frac{\partial^2 w_2}{\partial t^2} + c^2\frac{\partial^4 w_2}{\partial y^4} = -y\ddot{\varphi} - g\cos\varphi \tag{7}$$

with boundary and initial conditions

$$w_2(t,0) = 0, \quad \frac{\partial w_2(t,0)}{\partial y} = 0$$

$$\frac{\partial^2 w_2(t,l)}{\partial y^2} = 0, \quad \frac{\partial^3 w_2(t,l)}{\partial y^3} = 0 \tag{8}$$

$$w_2(0,y) = 0, \quad \dot{w}_2(0,y) = 0 \tag{9}$$

Here

$$\varphi(t) = \frac{\partial w_1(t,y')}{\partial y}, \quad y' = \begin{cases} vt - 2kl & \text{when } t > 2kl/v \\ 2(k+1)l - vt & \text{when } t < 2(k+1)l/v \end{cases}$$

$k = 0, 1, \ldots$

(υ is the propagation velocity of flexural waves along the rod [3]).

Boundary-value problem (7)–(9) can be solved in a similar way.

We obtain the following differential equation for the coefficients $\theta_m(t)$

$$\ddot{\theta}_m^1(t) + k_m^2\theta_m^1(t) = \Phi_m^1[t] \tag{10}$$

$$\Phi_m^1[t] = -\frac{1}{\alpha_m^2}\int_0^l (y\ddot{\varphi} + g\cos\varphi)Z_m(y)dy$$

(the square brackets $[t]$ denote that the function $\Phi^1{}_m$ depends implicitly on time).

Constructing the solution of Eq. (10) with initial conditions (9) we conclude that the elastic vibrations of the rod during motion are given by the formula

$$w_2(t,y) = \sum_{m=1}^{\infty}\left[\frac{1}{k_m}\int_0^t \Phi_m^1[\tau]\sin k_m(t-\tau)d\tau\right]Z_m(y) \tag{11}$$

Solution (11), after reduction, can be represented in the form of the sum of forced and free vibrations, due to the perturbing forces $\Phi_m^1[t]$.

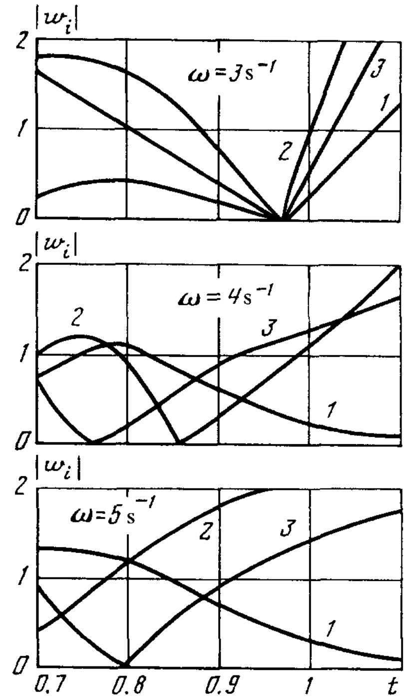

Fig. 1.

Due to various forms of resistance, the free vibrations will gradually decay and only the forced vibrations are of practical value. Omitting the details, we will represent the forced vibrations (ignoring the inherent weight) when $t > 2kl/\upsilon$, $(k = 0, 1, \ldots)$ from general solution (11) in the form

$$w_2(t, y) = \sum_{m=1}^{\infty} \frac{A\omega^2\gamma_m^2\beta_m\bar{Z}_m(y)}{\lambda_m^2(\omega^2 - k_m^2)}[\omega_{m1}(a_m)\sin\omega t + \theta_{m2}(a_m)\cos\omega t] \tag{12}$$

$$\theta_{m1}(a_m) = l_m^+(\operatorname{sh} a_m - \beta_m \operatorname{ch} a_m)(b_m^- d_m^- / 2 + 2\eta_m^2\omega^2) +$$

$$+ l_m^-(\sin a_m + \beta_m \cos a_m)(b_m^+ d_m^+ / 2 - 2\eta_m^2\omega^2)$$

$$\theta_{m2}(a_m) = \eta_m k_m^2 \omega 2[l_m^-(\cos a_m - \beta_m \sin a_m) + l_m^+(\operatorname{ch} a_m - \beta_m \operatorname{sh} a_m)]$$

$$l_m^{\pm} = \{[\eta_m^2 \pm (\omega + k_m)^2][\eta_m^2 \pm (\omega - k_m)^2]\}^{-1}$$

$$d_m^{\pm} = \eta_m^2 \pm (\omega^2 - k_m^2), \quad b_m^{\pm} = \eta_m^2 \pm \omega^2, \quad \eta_m = \lambda_m \nu$$

$$\upsilon = 2\sqrt{\omega c}, \quad a_m = \lambda_m(\upsilon t - 2kl)$$

The forced vibrations when $t < 2(k+1)l/\upsilon$ can be obtained using (12) by replacing k and υ by $-(k+1)$ and $-\upsilon$. When the denominator of the mth terms of series (12) becomes equal to zero, the frequency of the perturbing force will be approximately equal to one of the values of k_m, $(3 \pm 2\sqrt{2})k_m$ $(m = 1, 2, \ldots)$, which are found from the condition $(l_m^-)^{-1} = 0$. In this case we obtain a state of resonance.

Comparing problem (1)–(3) with problem (7)–(9) we obtain new values of $(3 \pm 2\sqrt{2})k_m$ for the state of resonance.

To compare the two types of forced vibrations described by (5) and (12) we carried out numerical calculations for two modes of vibration of a steel rod of circular cross-section as a function of time for various values of the frequency ω.

The results are shown in the figure for $y = 700$ cm. Curves 1 and 2 represent the behaviour of $|w_1|$ and $|w_2|$, i.e. the absolute values of the forced vibrations of the rod, calculated from (5) and (12) respectively. Curve 3 represents $|w_3| = |w_1 + w_2|$. It can be seen that the values of the buckling $|w_2|$ is higher than the bucklings $|w_1|$ and $|w_2|$, and value of $|w_i|$ initially falls and then increases sharply as t increases. The greatest absolute value of $|w_2|$ is 3.14 times greater than $|w_1|$ and 1.47 times greater than $|w_3|$.

In the calculations using formulae (5) and (12) we took the first six terms of the series for the following values of the parameters: $A = 1$ cm, $l = 700$ cm, $E = 2 \times 10^6$ kg/cm^2, $\rho = 7.8 \times 10^{-3}$ kg/cm^3, external diameter of the rod $d_1 = 30$ cm and internal diameter of the rod $d_2 = 28.6$ cm.

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Translated by R.C.G.